

Here $A^{x-x'} \neq 1$, otherwise $A^x = A^{x'}$, which is impossible, since $x \neq x'$. With this remark, formula (5) establishes our lemma.

4. The proof of the theorem stated in section 2 now follows immediately. The set of all elements A^s such that $B^{-1}A^sB = A^t$ for some t evidently forms a group N , as we see by multiplying together any two relations of this form. By the lemma, $N \neq \{1\}$. The elements A^t form a group N' conjugate to N in G . But, since N and N' , as conjugate groups, have the same order, and each is a subgroup of the cyclic group A , it follows that $N' = N$, i.e., $B^{-1}NB = N$.

Thus N is invariant under conjugation by B . As subgroup of $\{A\}$, N is obviously invariant under conjugation by A ; hence, N is invariant under conjugation by $A^x B^y$ for all x, y ; i.e., N is invariant in G .

This completes the proof of the theorem of section 2, and therewith the supersolvability of G .

¹ This concept was introduced by G. Zappa, "Sui gruppi supersolubili," *Rend. Sem. Math. Univ. Roma* (4), 2, 323-330 (1937).

² Huppert, B., "Über das Produkt von paarweise vertauschbaren zyklischen Gruppen," *Math. Zeitschr.*, 58, 243-264 (1953), p. 257.

³ Douglas, J., "On finite groups with two independent generators," these PROCEEDINGS, 37, 604-610, 677-691, 749-760, 808-813 (1951).

A SIMPLIFIED PROOF OF VON NEUMANN'S COORDINATIZATION THEOREM

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1. *Introduction.*—In 1936-37, von Neumann proved the following result: Suppose L is a complemented modular lattice possessing a homogeneous basis a_1, \dots, a_n with n finite and ≥ 4 . Then there exists a regular associative ring S with unit such that L is lattice-isomorphic to $L(S^n)$, where S^n denotes the set of vectors $(\alpha^1, \dots, \alpha^n)$ with all α^i in S and $L(S^n)$ denotes the set of finitely generated submodules of S^n , ordered by set inclusion (submodule means left submodule with S as ring of scalars).¹

Von Neumann defines S to be the auxiliary ring of L -numbers (α is in S means: $\alpha = (\alpha_{ij})$ for a certain family of lattice elements α_{ij} , each a relative complement of a_j in $a_i + a_j$). Then, by induction on m ($m = 1, \dots, n$), von Neumann shows:

$$\text{The sublattice } [0, a_1 + \dots + a_m] \text{ is isomorphic to } L(S^m). \quad (1)$$

Von Neumann's proof of (1) employs powerful and involved technique. We now point out that for $m = 1$, (1) is an immediate consequence of the regularity of S ; for $m = 2, 3$, (1) can be verified by straightforward calculations; and what is surprising, for $m > 3$, the validity of (1) can be deduced from its validity for $m = 3$ by simple lattice calculations.

Our simplification² of the proof of (1) covers extensions of von Neumann's

theorem to the case $n = 3$ supplemented by a Desarguesian³ (or Moufang⁴ or even weaker⁵) condition. With minor changes, it yields at once an extension wider than that of Jónsson.⁶ Jónsson assumes that L has a basis a_1, \dots, a_n with each a_i perspective to a subelement \bar{a}_i of a_1 and $\bar{a}_i = a_1$ for $i = 2, 3, 4$, (or for $i = 2, 3$ together with a supplementary condition). With non-Desarguesian hypotheses on L , the associative law for multiplication and the left distributive law may fail to hold in S .

For the special case that L is an $n - 1$ dimensional projective geometry and the a_i are points (then the only idempotents in S are $0, 1$), this gives a direct and rather transparent demonstration of the classical coordinatization theorem for projective geometry.

We sketch below our simplified proof of von Neumann's original theorem ($n \geq 4$). A detailed discussion will be given elsewhere.

2. *Notation.*— $A^i \equiv a_1 + \dots + a_i$, $A^0 \equiv 0$, $A_j^i \equiv a_1 + \dots + a_{j-1} + a_{j+1} + \dots + a_i$. An x is called an i -element if $x \leq A^i$ and $xA^{i-1} = 0$. If $v = (\alpha^1, \dots, \alpha^n)$ with $\alpha^j = 0$ for $j > i$ and with $\alpha^i = \text{idempotent}$ and $\alpha^i \alpha^j = \alpha^j$ for $j < i$, then v is called an i -vector. $M(w, \dots)$ denotes the submodule generated by the vectors w, \dots . If α is in S , the reach of α into a_j , written $(\alpha)_{j'}^r$, means $(\alpha_{ij} + a_i)a_j$ for any $i \neq j$.

Suppose x is an i -element. We choose an idempotent $e \equiv e(x)$ with $(e)_i^r = (x + A^{i-1})a_i$. For $j < i$, we define x^j to be $(x + A_j^{i-1})(a_i + a_j)$ and we choose any β^j in S with $(\beta^j)_{ij} \geq x^j$. Then $v(x)$ is defined to be the i -vector $(-e\beta^1, \dots, -e\beta^{i-1}, e, 0, \dots, 0)$. $M(v(x))$ is determined uniquely by x although $v(x)$ depends on the choice of $e(x)$.

If x is an arbitrary element in L , we choose a base-decomposition $x = x_1 + \dots + x_n$ with each x_i an i -element and so obtain a submodule $M(x) \equiv M(v(x_1), \dots, v(x_n))$. This $M(x)$ is determined uniquely by the particular choice of x_1, \dots, x_n , but it is not yet clear that it is the same for all base-decompositions of the given x .

3. *Statement of the Problem.*—It is easy to show that every element in $L(S^n)$ occurs among the $M(v(x_1), \dots, v(x_n))$ as x varies over L . The correspondence $x \rightarrow M(x)$ is therefore a lattice isomorphism of L onto $L(S^n)$ if the following statement holds: whenever x, y have base-decompositions $x_1 + \dots + x_n, y_1 + \dots + y_n$ respectively, then

$$x \leq y \text{ implies } M(v(x_1), \dots, v(x_n)) \leq M(v(y_1), \dots, v(y_n)), \quad (2)$$

$$M(v(x_1), \dots, v(x_n)) \leq M(v(y_1), \dots, v(y_n)) \text{ implies } x \leq y. \quad (3)$$

4. *Reduction of the Problem.*—In (2) and (3), we need only consider the case: $x = x_m$ for some $m \leq n$ and $y_j = 0$ for $j > m$. Then we may assume that $e(x_m) = e(y_m)$ since (i) $x_m \leq y_m$ implies that $v(x_m) = e(x_m)v(y_m)$ and (ii) $ee(y_m) = e(\text{idempotent})$ implies that $ev(y_m) = v(x_m)$ for some m -element $x_m \leq y_m$. Finally, we may even assume $e(x_m) = e(y_m) = 1$.

This settles the case $m = 1$. If $m \geq 2$, it is sufficient to verify (2), (3) when $y_j \neq 0$ for precisely one $j < m$ and we may even suppose that the exceptional j is $m - 1$. Thus we need only show that if

$$v(x_m) = (-\alpha^{m,1}, \dots, -\alpha^{m,m-1}, 1, 0, \dots, 0),$$

$$v(y_m) = (-\beta^{m,1}, \dots, -\beta^{m,m-1}, 1, 0, \dots, 0),$$

and $v(y_{m-1}) = (-\gamma^{m-1,1}, \dots, -\gamma^{m-1, m-2}, f, 0, 0, \dots, 0)$,
then,

$$x_m \leq y_{m-1} + y_m \text{ implies } v(x_m) = v(y_m) + \delta v(y_{m-1}) \text{ for some } \delta \text{ in } S, \quad (4)$$

$$\text{and } v(x_m) = v(y_m) + \delta v(y_{m-1}) \text{ for some } \delta \text{ in } S \text{ implies } x_m \leq y_{m-1} + y_m. \quad (5)$$

5. *The Case $m = 2$.*—The relation $v(x_2) = v(y_2) + \delta v(y_1)$ is equivalent in turn to each of: (i) $\beta^{2,1} - \alpha^{2,1}$ is in the left ideal of f , (ii) $(\beta^{2,1} - \alpha^{2,1})_1^r \leq (f)_1^r$, (iii) $((\beta^{2,1})_{2,1} + (\alpha^{2,1})_{2,1})a_1 \leq y_1$, (iv) $(x_2 + y_2)a_1 \leq y_1$, and (v) $x_2 \leq y_1 + y_2$. This settles the case $m = 2$.

6. *Deduction of (4) from (5).*—Assume $x_m \leq y_{m-1} + y_m$. Then, by an argument like that of paragraph 5, $\beta^{m,m-1} - \alpha^{m,m-1} = \delta f$ for some δ . If now we assume (5) holds, we obtain with this δ : $v(y_m) + \delta v(y_{m-1}) = v(\bar{x}_m)$ for some $\bar{x}_m \leq y_{m-1} + y_m$. Then,

$$x_m + A^{m-2} = \bar{x}_m + A^{m-2};$$

$$\text{so, } x_m = (y_{m-1} + y_m)(x_m + A^{m-2}) = \bar{x}_m,$$

and hence (4) holds.

7. *Reduction of (5) to the Case $m = 3$.*—To complete the proof, we need only prove (5) for the case $m \geq 3$.

If $m > 3$ and $v(x_m) = v(y_m) + \delta v(y_{m-1})$ we choose any $j \leq m - 2$ and “project” into $[0, a_j + a_{m-1} + a_m]$. Now if (5) is assumed for the special case $m = 3$ we can obtain

$$x_m \leq A_j^{m-2} + y_{m-1} + y_m \text{ for all } j \leq m - 2;$$

hence $x_m \leq y_{m-1} + y_m$, proving (5). So we need only prove (5) for the case $m = 3$.

8. *Proof of (5) for the Case $m = 3$.*—Suppose $v(x_3) = v(y_3) + \delta v(y_2)$. Then, by an argument like that of paragraph 5: $x_3 \leq a_1 + y_2 + y_3$. If now we assume (5) holds for the special case $m = 3, f = 1$, we have also

$$x_3 \leq (f_{21}a_2 + y_2) + y_3,$$

$$\text{and so, } x_3 \leq (a_1 + y_2 + y_3)(f_{21}a_2 + y_2 + y_3) = y_2 + y_3.$$

So we need only prove (5) for the case $m = 3, f = 1$. Now we have

$$(\beta^{3,2} - \alpha^{3,2})(-\gamma^{2,1}) = \beta^{3,1} - \alpha^{3,1},$$

$$(\beta^{3,2}\gamma^{2,1} + \beta^{3,1})_{3,1} = (\alpha^{3,2}\gamma^{2,1} + \alpha^{3,1})_{3,1}.$$

If we calculate the right and left sides of the last equality using the ternary relation⁷

$$(\beta\gamma + \delta)_{3,1} = (\gamma_{2,1} + (\delta_{3,1} + a_2)(\beta_{3,2} + a_1))(a_3 + a_1),$$

we obtain $y_2 + y_3$ and $y_2 + x_3$ respectively. Hence, $x_3 \leq y_2 + y_3$, verifying (5). This completes the proof of von Neumann's theorem.

¹ See von Neumann, *Continuous Geometry* (Princeton University Press, 1960), part 2, chapters 1–14, especially Theorem 14.1 on page 208, or Fryer, K. D., and I. Halperin, “On the coordinatization theorem of J. von Neumann,” *Can. Journ. of Math.*, **7**, 432–444 (1955), or Amemiya, I., “On the representation of complemented modular lattices,” *Journ. Math. Soc. Jap.*, **9**, 263–279 (1957).

² For simplifications in the construction of S , see Fryer, K. D., and I. Halperin, "Coordinates in geometry," *Trans. Roy. Soc. Can.*, **48**, Ser. 3, 11–26 (1954).

³ Fryer, K. D., and I. Halperin, "The von Neumann coordinatization theorem for complemented modular lattices," *Acta Sci. Szeged*, **17**, 203–249 (1956).

⁴ Fryer, K. D., and I. Halperin, "On the construction of coordinates for non-Desarguesian complemented modular lattices," *Proc. Roy. Neth. Acad. (Amsterdam)*, **61**, 142–161 (1958); Amemiya, I., and I. Halperin, "On the coordinatization of complemented modular lattices," *Proc. Roy. Neth. Acad. (Amsterdam)*, **62**, 72–78 (1959) and "Complemented modular lattices derived from non-associative rings," *Acta Sci. Szeged*, **20**, 181–201 (1959).

⁵ Baer, R., "Homogeneity of projective planes," *Amer. J. Math.*, **64**, 137–152 (1942); Fryer, K. D., "Coordinates in non-Desarguesian complemented modular lattices," *Proceedings of Symposia in Pure Mathematics, volume II; Lattice Theory* (Providence: American Mathematical Society, 1961), pp. 71–77.

⁶ Jónsson, Bjarni, "Representations of complemented modular lattices," *Trans. Amer. Math. Soc.*, **97**, 64–94 (1960).

⁷ See (5.2.2) in ref. 3 above.

COLICINE V*

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Nearly a decade ago this laboratory began an investigation on the nature of colicines—antibacterial agents of remarkable potency which are elaborated by certain *Enterobacteriaceae* and which kill other microorganisms of the same species. These agents were first described by Gratia in 1925¹ who observed that cell-free filtrates of a virulent strain of colon bacillus with which he was working contained an agent, "principle V," which killed a sensitive host strain "coli ϕ ." Gratia observed that his principle was thermostable and that it diffused through cellophane membranes. He pointed out that the agent bore a remarkable similarity to bacteriophage, yet differed in that it would not reduplicate upon serial passage.

A resurgence of interest in the colicines took place in the mid-forties, largely through the work of Frédéricq,² to whom most of our modern knowledge concerning the distribution, specificity, and tenuous relationship of the colicines to the bacteriophages can be attributed.³ Extensive as our knowledge is in this regard, our understanding of their chemical nature has remained singularly enigmatic despite the efforts of a number of investigators.⁴

Several years ago we in this laboratory described the isolation of one of the colicines—colicine K.⁵ This substance proved to be a lipocarbohydrate-protein complex, identical with the somatic O antigen of the microorganism from which it was derived. The material had exceedingly potent antibacterial properties. It was antigenic in rabbits and the antisera specifically precipitated the colicine and neutralized its antibacterial properties as well.⁶ Although the colicines and bacteriophages show striking resemblances,² we were unable to demonstrate any serological relationship between our colicine K and the coli-dysentery phage T6, the virus to which this colicine is presumed to be related.

We have now isolated another and different colicine—colicine V. We chose to